ACTIONS OF BAUMSLAG-SOLITAR GROUPS ON SURFACES.

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ABSTRACT. Let $BS(1,n) =< a,b \mid aba^{-1} = b^n >$ be the solvable Baumslag-Solitar group, where $n \geq 2$. It is known that $BS(1,n)$ is isomorphic to the group generated by the two affine maps of the real line: $f_0(x) = x+1$ and $h_0(x) = nx$.

This paper deals with the dynamics of actions of $BS(1,n)$ on closed orientable surfaces. We exhibit a smooth $BS(1,n)$ action without finite orbits on $\mathbb{T}^2$, we study the dynamical behavior of it and of its C^1-pertubations and we prove that it is not locally rigid.

We develop a general dynamical study for faithful topological $BS(1,n)$-actions on closed surfaces S. We prove that such actions $< f,h \mid h\circ f\circ h^{-1} = f^n >$ admit a minimal set included in $fix(f)$, the set of fixed points of f, provided that $fix(f)$ is not empty.

When $S = \mathbb{T}^2$, we show that there exists a positive integer N, such that $fix(f^N)$ is non-empty and contains a minimal set of the action. As a corollary, we get that there are no minimal faithful topological actions of $BS(1,n)$ on $\mathbb{T}^2$.

When the surface S has genus at least 2, is closed and orientable, and f is isotopic to identity, then $fix(f)$ is non empty and contains a minimal set of the action. Moreover if the action is C^1 then $fix(f)$ contains any minimal set.

1. Introduction and statements

An important question on group actions is existence and stability of global fixed points. For Lie group actions, it was shown by Lima [Lim64] that any action of the abelian Lie group $\mathbb{R}^n$ on a surface with non-zero Euler characteristic has a global fixed point. This result was later extended by Plante [Pla86] to nilpotent Lie groups. On the other hand, Lima [Lim64] and Plante[Pla86] proved that the solvable Lie group $GA(1,\mathbb{R})$ acts without fixed points on every compact surface.

For discrete group actions, Bonatti [Bon89] showed that any $\mathbb{Z}^n$ action on surfaces with non-zero Euler characteristic generated by diffeomorphisms C^1 close to the identity has a global fixed point. Druck, Fang and Firmo [DF02] proved a discrete version of Plante's theorem.

This paper deals with the dynamics of actions of the solvable Baumslag-Solitar group, $BS(1,n) =< a,b \mid aba^{-1} = b^n >$, where $n \geq 2$, on closed surfaces.

It is well known that $BS(1,n)$ has many actions on $\mathbb{R}$. The standard action on $\mathbb{R}$ is the action generated by the two affine maps $f_0(x) = x+1$ and $h_0(x) = nx$ (where $f_0 \equiv b$ and $h_0 \equiv a$).

Actions of solvable groups on one-manifolds have been studied by Plante [Pla86], Ghys [Ghy01], Navas [Nav04], Farb and Franks[FF01], Moriyama[Mor94] and Rebelo and

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Silva[RS03]. In [FF01], as a corollary of a result of M. Shub [Shu70] on expanding maps, they showed the following:

Theorem. *(Farb-Franks-Shub).*
There are neighborhoods of f_0 and h_0 in the uniform C^1 topology such that whenever f and h are chosen from these respective neighborhoods and the group generated by $\{f, h\}$ is isomorphic to $BS(1, n)$ then the perturbed action is topologically conjugate to the original action.

In contrast, Hirsch (see [Hir75]) has found analytic actions of $BS(1, n)$ on $\mathbb{R}$ which are not topologically conjugate to the standard action (but they are semiconjugate).

Definition 1.1. *The* **standard** $BS(1, n)$**-action on** $S^1 = \mathbb{R} \cup \infty$ *is the action generated by the two Moebius maps $f_0(x) = x + 1$ and $h_0(x) = nx$. It has a global fixed point at ∞.*

Farb-Franks-Shub Theorem remains true for the standard $BS(1, n)$-action on S^1, since a faithful BS-action C^1 close to the standard action on S^1 always has a global fixed point (the proof of this fact is analoguous to the proof of lemma 6.1) and therefore it can be seen as an action on $\mathbb{R}$.

More recently, L. Burslem and A. Wilkinson [BW04] gave a classification (up to conjugacy) of real analytic actions of $BS(1, n)$ on S^1. In particular, they proved that every representation of $BS(1, n)$ into $\text{Diff}^\omega(S^1)$ is C^∞-locally rigid and that for each $r \geq 3$ there are analytic actions of $BS(1, n)$ that are C^r-locally rigid, but not C^{r-1}-locally rigid (for the definition of local rigidity see section 2).

This results are proved by using a dynamical approach. The dynamics of C^2 $BS(1, n)$-actions on S^1 is now well understood, due to Navas work on solvable groups of circle diffeomorphisms (see [Nav04]). In particular, Burslem and Wilkinson (see [BW04]) proved that any C^2 faithful $BS(1, n)$-action on S^1 admits a finite orbit. Recently we have extended this result to C^1 case (see [GL11]). Also, we have proved that any C^1 faithful $BS(1, n)$-action on S^1 is semiconjugated (up to passing to a finite index subgroup) to the standard one.

The dynamical situation of $BS(1, n)$-actions on closed surface (even on $\mathbb{T}^2$) is more complicated. Non trivial examples of $BS(1, n)$ actions on closed surfaces can be constructed using actions of the affine real group $GA(1, \mathbb{R}) := \{x \mapsto \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}, \ \alpha > 0\}$. Actions of $GA(1, \mathbb{R})$ on closed surfaces have been studied by Plante-Thurston [PT76], Plante [Pla86], Belliart-Liousse [BL94].

On the other hand, C^1 faithful $BS(1, n)$-action on $\mathbb{T}^2$ can be constructed by using products of smooth actions on S^1. In the case where the circle actions are both C^1 and faithful, finite orbits always exist. But, faithful smooth $BS(1, n)$-actions on $\mathbb{T}^2$ can be obtained as the product of a faithful $BS(1, n)$-action and a non faithful one. In this case, finite orbits may not exist.

An important family of such examples is given by $< f_0, h_k >$, where :

$$f_0(x, \theta) = (x + 1, \theta) \text{ and } h_k(x, \theta) = (nx, k(\theta)),$$

where $x \in \mathbb{R} \cup \infty$, $\theta \in S^1$ and k is any (orientation preserving) circle homeomorphism.

In section 3, we explain the construction of these examples and exhibit a faithful smooth action of $BS(1,n)$ on $\mathbb{T}^2$ without finite orbits that can be considered as "the standard BS-action" on $\mathbb{T}^2$. More precisely,

Definition 1.2. *The* **standard** $BS(1,n)$**-action on** $\mathbb{T}^2$ *is the action generated by :*

$$f_0(x,\theta) = (x+1,\theta)$$

and

$$h_0(x,\theta) = (nx, \ln(n)+\theta),$$

where $x \in \mathbb{R} \cup \infty$ *and* $\theta \in S^1$.

Our first result is the following:

Theorem 1.
The group $< f_0, h_k >$ *generated by* f_0 *and* h_k *is isomorphic to* $BS(1,n)$.
If the rotation number of k *is rational, there exist finite BS-orbits.*
If the rotation number of k *is irrational, there are no finite BS-orbits and the unique minimal set for the BS-action is included in* $\infty \times S^1 = fix(f_0)$.

Corollary 1. *There exist* C^∞ *faithful BS-actions arbitrary* C^∞*-close to the standard torus BS-action* $< f_0, h_0 >$ *that are not topologically conjugate to* $< f_0, h_0 >$.

This implies that the standard BS-action on $\mathbb{T}^2$ does not satisfy the rigidity properties described in the Farb-Franks-Shub theorem for the standard BS-action on S^1.

This property can also be compared to the rigidity result recently proved by Mc Carthy : *"The trivial* $BS(1,n)$*-action on a compact manifold does not admit* C^1 *faithful perturbations"* (see [MC10]).

Then we consider perturbed actions of the standard one. In particular, we prove that there exists either a finite orbit or a unique minimal set. Recall that a **minimal set** for an action of a group G on a compact metric space X is a non-empty closed G-invariant subset of X such that if $K \subset M$ is a closed G-invariant set then either $K = M$ or $K = \emptyset$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the circles defined by $\mathcal{C}_1 = \infty \times S^1$ and $\mathcal{C}_2 = 0 \times S^1$. Note that both circles are h_0-invariant.

Theorem 2. *Let us consider a BS-action* $< f, h >$ *on* $\mathbb{T}^2$ *generated by* f *and* h *sufficiently* C^1*-close to* f_0 *and* h_0 *respectively. Then:*

1. *there exists two circles* $\mathcal{C}'_1$ *and* $\mathcal{C}'_2$ *close to* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *respectively which are* h*-invariant. Moreover, the* ω_h*-limit set of any point in* $\mathbb{T}^2 \setminus \mathcal{C}'_2$ *is included in* $\mathcal{C}'_1$ *and the* α_h*-limit set of any point in* $\mathbb{T}^2 \setminus \mathcal{C}'_1$ *is included* $\mathcal{C}'_2$.
2. *the set of* f*-fixed points is not empty and it is contained in the circle* $\mathcal{C}'_1$.
3. *either :*
   (a) *there exist finite BS-orbits contained in* $\mathcal{C}'_1$, *or*
   (b) *the action has a unique minimal set* M *which is included in* $\mathcal{C}'_1$ *(and in the set of* f*-fixed points). Moreover,* M *is either* $\mathcal{C}'_1$ *or a Cantor set.*

We check that the "standard action" on $\mathbb{T}^2$ satisfies item 3(b) but in the proof of Corollary 1 we exhibit C^∞-perturbations of it that have a different dynamical behavior: they satisfy item 3(a). In section 6, we exhibit an example of an action with a C^1 persistent global fixed point. More precisely, we construct an action with fixed point satisfying that any C^1-perturbation of it also has fixed point.

On the other hand, we develop a general dynamical study for faithful $BS(1,n)$-actions on closed surfaces. From now on, let us consider f and h two homeomorphisms that generate a $BS(1,n)$-action, that is, $h \circ f \circ h^{-1} = f^n$.

Our first "dynamical" result on the torus concerns the rotation set of f (for the definition see Section 2).

Theorem 3. *Let $< f, h >$ be a faithful action of $BS(1,n)$ on $\mathbb{T}^2$. Then there exists a positive integer N, such that f^N is isotopic to identity and has a lift whose rotation set is the single point $\{(0,0)\}$. Moreover, the set of f^N-fixed points denoted by $fix(f^N)$ is non-empty.*

Remark 1.1. *In section 3, we exhibit two diffeomorphisms F and H generating a faithful action of $BS(1,n)$ on $\mathbb{T}^2$, where F admits periodic orbits but it does not have fixed points.*

Since the group $< f^N, h >$ is isomorphic to $BS(1, nN)$, Theorem 3 allows us to restrict our study on the torus to the case where f is isotopic to identity, the rotation set of a lift of f is $\{(0,0)\}$ and f has fixed points. In this situation we prove that there exists a BS-minimal set included in the set of f-fixed points. More precisely, we prove the more general following statement.

Theorem 4. *Let X be a compact metric space and $< f, h >$ be a representation of $BS(1,n)$ in $Homeo(X)$.*
(a) If $fix(f)$ is non-empty, then:

1. *If $x_0 \in fix(f)$ then $\alpha_h(x_0)$ is contained in $fix(f)$.*
2. *There exists an BS-minimal set included in $fix(f)$. Moreover, this BS-minimal set coincides with a h-minimal set in $fix(f)$.*
3. *If the set of f-fixed points is finite then the action admits a global finite orbit.*
4. *Let $\mathcal{M}$ be an BS-minimal set satisfying $\mathcal{M} \cap fix(f) \neq \emptyset$, then $\mathcal{M} \subset fix(f)$.*

(b) If the set of periodic points of f, $per(f)$, is non-empty, then there exist a positive integer N and a BS-minimal set, M, such that $M \subset fix f^N$.

As a consequence of item (b) of Theorem 4, Theorem 3 and the fact that $< f, h >$ is a faithful representation of $BS(1,n)$, we have the following:

Corollary 2. *Let X be a compact metric space and $< f, h >$ be a faithful representation of $BS(1,n)$ in $Homeo(X)$ such that $Per(f)$ is non-empty. Then the action of $< f, h >$ is not minimal.*
In particular:

1. *There is no faithful minimal action of $BS(1,n)$ by homeomorphisms on $\mathbb{T}^2$.*

(2) *Let Σ be a compact surface of non zero Euler characteristic. A faithful topological $BS(1,n)$-action $< f,h >$ on Σ is not minimal, provided that f is isotopic to identity.*

Remark 1.2. *Item (2) is a consequence of Theorem 4 (a)(2) and Lefschetz's fixed point theorem.*

When $< f,h >$ is a topological action of $BS(1,n)$ on a closed surface S satisfying that any f-invariant probability has support included in the set of f-fixed points, we prove the following:

Theorem 5. *Let S be a closed orientable surface and $< f,h >$ be a representation of $BS(1,n)$ in $Homeo(S)$. Suppose that for any f-invariant probability measure μ, supp$(\mu) \subset fix(f)$. Then:*

(1) *Any f-minimal set is a fixed point. The set of periodic points of f, per(f), coincides with the set $fix(f)$.*
(2) *Any BS-minimal set is included in $fix(f)$. Moreover, any BS-minimal set coincides with a h-minimal set in $fix(f)$.*
(3) *Topological entropy of f, $ent_{top}(f) = 0$.*

For next corollary, that we prove using Theorem 1.3 of [FH06], we need the following:

Definition 1.3. *Let $g \in Diff^1(S)$, an N-periodic point x_0 is called* **elliptic** *if the eigenvalues of the differential of g at x_0, $Dg^N(x_0)$, have module 1.*

Corollary 3. *Let S be a closed orientable surface and $< f,h >$ be a representation of $BS(1,n)$ in $Diff^1(S)$ such that:*

- *If S has genus at least 1, f is isotopic to identity.*
- *If $S = S^2$, some iterate of f has at least three fixed points.*

Then there exists a positive integer N such that :

(1) *Any f-minimal set is a periodic point. The set of periodic points of f, per(f), coincides with the set $fix(f^N)$.*
(2) *Any BS-minimal set is included in $fix(f^N)$. In fact, any BS-minimal set is included in a subset of f-elliptic points in $fix(f^N)$.*
(3) *$ent_{top}(f) = 0$.*

In addition, if S has genus at least 2, $N = 1$.

Finally, we have the following open questions:

(1) Does it exist a faithful action of $< f,h >= BS(1,n)$ on $\mathbb{T}^2$ with h non isotopic to identity? We know that there does not exist representation of $BS(1,n)$ into $Aff(\mathbb{T}^2) = SL(2,\mathbb{Z}) \ltimes \mathbb{R}^2$ the group consisting of maps :$g(x,y) = A.(x,y) + V$, where $A \in SL(2,\mathbb{Z})$ and $V \in \mathbb{R}^2$.
(2) Does it exist a faithful continuous action of $< f,h >= BS(1,n)$ on $\mathbb{T}^2$ with minimal sets outside *per*(f) ?
(3) Is the product action on $(\mathbb{R} \cup \infty) \times S^1$ generated by $f_0(x,\theta) = (x+1,\theta)$ and $h_0(x,\theta) = (nx, k(\theta))$, where k is a circle *north-south* diffeomorphism topologically rigid ?

In Section 2, we give definitions, properties and basic tools that we use in the rest of the paper. We exhibit examples of $BS(1,n)$ acting on $\mathbb{T}^2$, and Theorem 1 and Corollary 1 are proved in Section 3. The goal in Section 4 is proving Theorem 3. In Section 5 we prove Theorems 4, 5 and Corollaries 2 and 3. In Section 6, we consider perturbations of the standard $BS(1,n)$-action on $\mathbb{T}^2$: we describe their minimal sets by proving Theorem 2. We also construct an action with a persistent global fixed point.

2. Definitions-Notations

2.1. **Isotopy class of torus homeomorphims.**

We denote by $Homeo_{\mathbb{Z}^2}(\mathbb{R}^2)$ the set of homeomorphisms $F:\mathbb{R}^2\to\mathbb{R}^2$ such that $F(\mathbb{Z}^2)\subseteq\mathbb{Z}^2$ and $Homeo^0_{\mathbb{Z}^2}(\mathbb{R}^2)$ the set of homeomorphisms $F:\mathbb{R}^2\to\mathbb{R}^2$ such that $F(\mathbb{Z}^2)\subseteq\mathbb{Z}^2$ and $F(x+P)=F(x)+P$, for all $x\in\mathbb{R}^2$ and $P\in\mathbb{Z}^2$.

Note that, a lift of a 2-torus homeomorphism isotopic to identity belongs to $Homeo^0_{\mathbb{Z}^2}(\mathbb{R}^2)$. Conversely, if a 2-torus homeomorphism admits a lift $F\in Homeo^0_{\mathbb{Z}^2}(\mathbb{R}^2)$ then it is isotopic to identity.

Let $g:\mathbb{T}^2\to\mathbb{T}^2$ be a homeomorphism and let $G:\mathbb{R}^2\to\mathbb{R}^2$ be a lift of g. We can associate to G a linear map A_G defined by :

$$G(p+(m,n))=G(p)+A_G(m,n),\ \text{for any } m,n \text{ integers.}$$

By definition, it is clear that $G\in Homeo^0_{\mathbb{Z}^2}(\mathbb{R}^2)$ (that is g is isotopic to identity) if and only if $A_G=Id$.

This map satisfy the following properties :

(1) A_G does not depend neither on the integers m and n nor on the lift G of g. In fact, A_G is the morphism induced by g on the first homology group of $\mathbb{T}^2$. So we can also denote A_g for A_G and we will use both notations.
(2) $A_{G\circ F}=A_G\circ A_F$ and $A_{G^{-1}}=A_G^{-1}$,
(3) $A_G\in GL(2,\mathbb{Z})$, in particular
(4) $detA_G=+1$ or -1.

2.2. **Rotation set and rotation vectors.**

2.2.1. **Definitions.**

Let f be a 2-torus homeomorphism isotopic to identity. We denote by $\tilde f$ a lift of to $\mathbb{R}^2$. We call **$\tilde f$-rotation set** the subset of $\mathbb{R}^2$ defined by

$$\rho(\tilde f)=\bigcap_{i=1}^{\infty}\overline{\bigcup_{n\geq i}\left\{\frac{\tilde f^n(\widetilde x)-\widetilde x}{n},\quad \widetilde x\in\mathbb{R}^2\right\}}.$$

Equivalently, $(a,b)\in\rho(\tilde f)$ if and only if there exist sequences $(\widetilde{x_i})$ with $\widetilde{x_i}\in\mathbb{R}^2$ and $n_i\to\infty$ such that

$$(a,b)=\lim_{i\to\infty}\frac{\tilde f^{n_i}(\widetilde{x_i})-\widetilde{x_i}}{n_i}.$$

Let $\widetilde{x}$ be in $\mathbb{R}^2$. The **$\widetilde{f}$-rotation vector of** $\widetilde{x}$ is the 2-vector defined by $\rho(\widetilde{f},\widetilde{x}) = \lim_{n\to\infty} \dfrac{\widetilde{f}^n(\widetilde{x}) - \widetilde{x}}{n} \in \mathbb{R}^2$ if this limit exists.

From now on, we use both $\tilde{f}$ or F for a lift of f to $\mathbb{R}^2$.

2.2.2. **Some classical properties and results on the rotation set.** Let f be a 2-torus homeomorphism isotopic to the identity and $\widetilde{f}$ be a lift of f to $\mathbb{R}^2$.

- Let $\widetilde{x} \in \mathbb{R}^2$ be such that $\rho(\widetilde{f},\widetilde{x})$ exists. Then $\rho(\widetilde{f},\widetilde{x}) \in \rho(\widetilde{f})$.
- If $\widetilde{f}$ has a fixed point then $(0,0) \in \rho(\widetilde{f})$.
- Misiurewicz and Ziemian (see [MZ89]) have proved that:
  1. $\rho(\tilde{f}^n) = n\rho(\tilde{f})$
  2. $\rho(\tilde{f} + (p,q)) = \rho(\tilde{f}) + (p,q)$,
  3. the rotation set is a compact convex subset of $\mathbb{R}^2$.

2.3. **C^r-local rigidity, where $r \in \mathbb{N} \cup \{\infty, \omega\}$.**

Definition 2.1. *An action $< f_1, h_1 >$ of $BS(1,n)$ on a smooth manifold is* **C^r-locally rigid** *($r \in \mathbb{N}\cup\infty\cup\omega$) if there exist neighborhoods of f_1 and h_1 in the C^1-topology such that whenever f and h are C^r maps chosen from these neighborhoods and the group generated by $< f, h >$ is isomorphic to $BS(1,n)$, then the perturbed action is C^r conjugate to the original one, that is there exists a C^r-diffeomorphism H such that $H \circ f \circ H^{-1} = f_1$ and $H \circ h \circ H^{-1} = h_1$.*

2.4. **$BS(1,n)$-actions.** Consequence of the conjugation between f^n and f.

As consequences of the group-relation $h \circ f \circ h^{-1} = f^n$, we get easily the following two propositions :

Proposition 2.1. *Let f and h be homeomorphisms satisfying $h \circ f \circ h^{-1} = f^n$, then*
1. *$h \circ f^p \circ h^{-1} = f^{np}$, for all integer p,*
2. *$h^p \circ f \circ h^{-p} = f^{n^p}$, for all positive integer p.*

Proposition 2.2.
Let f and h be as in the previous proposition, then
1. *$h(fix(f)) = fix(f^n)$,*
2. *Let $per(f)$ be the set of periodic points of f, then $h(per(f)) = per(f^n)$. More precisely, if x is an f^p fixed point then $h(x)$ is an $(f^n)^p = f^{np}$ fixed point.*
3. *If M_f is an f-minimal set then $h(M_f)$ is a minimal set of f^n.*
4. *Let $ent(f)$ be the topological entropy of f. Then $ent(f)$ is 0 or ∞.*

Proof of (4). Since $ent(f^n) = n.ent(f)$ and $ent(h \circ f \circ h^{-1}) = ent(f)$ the possible values for $ent(f)$ are 0 or ∞.

3. Examples of $BS(1,n)$-actions on $\mathbb{T}^2$

In this section we will exhibit examples of $BS(1,n)$-actions on $\mathbb{T}^2$.

3.1. **Product of faithful actions on S^1.**

Let $< f_i, h_i >$, $i = 1, 2$ be two C^1 actions of $BS(1,n)$ on S^1, we construct an action of $BS(1,n)$ on $\mathbb{T}^2$ by setting : $f = (f_1, f_2)$ and $h = (h_1, h_2)$. Clearly, the $< f, h >$-orbit of a point $x = (x_1, x_2)$ in $\mathbb{T}^2$ is the product of the $< f_1, h_1 >$-orbit of x_1 and the $< f_2, h_2 >$-orbit of x_2.

According to [GL11], there exists a finite $< f_i, h_i >$-orbit at some point $y_i \in S^1$, hence the $< f, h >$-orbit of the point $y = (y_1, y_2)$ is finite.

The following two sections show examples of BS-actions on $\mathbb{T}^2$ without finite orbits.

3.2. **Product of non faithful actions on S^1.**

We construct faithful $BS(1,n)$-actions without finite orbits as product of a faithful circle action and a non faithful one.

Let $< f_1, h_1 >$ be a faithful action of $BS(1,n)$ on S^1 and k be a circle homeomorphism. We construct a faithful action of $BS(1,n)$ on $\mathbb{T}^2$ by setting: $f = (f_1, Id)$ and $h = (h_1, k)$. Clearly, if k has no finite orbit, there is no global finite orbit.

3.3. **Actions that come from actions of the affine group of the real line.**

3.3.1. **Actions of $GA(1,\mathbb{R})$ and induced $BS(1,n)$-actions on the circle.**

Identifying the affine real map $x \mapsto ax + b$ with (a, b), the affine group of the real line, $GA(1,\mathbb{R})$, is the group $\mathbb{R}_{>0} \times \mathbb{R}$ endowed with the product $(a, b) \times (a', b') = (aa', ab' + b)$.

The Baumslag-Solitar group $BS(1,n)$ can be seen as the subgroup generated by the elements $(1, 1)$ and $(n, 0)$.

Let $\Phi : GA(1,\mathbb{R}) \to Diff^r(M)$ be an action of $GA(1,\mathbb{R})$, the **induced** $BS(1,n)$**-action** is the restriction of Φ to $< (1, 1), (n, 0) >$.

The standard actions.

Definition 3.1. *The* **standard action of** $GA(1,\mathbb{R})$ **on the circle** *is the action by Moebius maps on the projective line, that is :*

$$\Phi^{stand} : \begin{cases} GA(1,\mathbb{R}) & \to Diff_+^{\omega}(S^1) \\ (a,b) & \mapsto \Phi^{stand}_{(a,b)} \end{cases},$$

where

$$\Phi^{stand}_{(a,b)} : \begin{cases} \mathbb{R} \cup \{\infty\} & \to \mathbb{R} \cup \{\infty\} \\ x & \mapsto ax + b \end{cases}$$

This action is faithful and has a global fixed point at ∞.

Definition 3.2. *The* **standard action of** $BS(1,n)$ **on the circle** *is the induced* $BS(1,n)$*-action, it is generated by the two Moebius maps* $f_0(x) = \Phi^{stand}_{(1,1)}(x) = x + 1$ *and* $h_0(x) = \Phi^{stand}_{(n,0)}(x) = nx$.

It is faithful and has a global fixed point at ∞. Moreover f_0 has a unique fixed point at ∞ that is elliptic and h_0 has two hyperbolic fixed points : ∞ that is an attractor and 0 that is a repeller.

The orbit of a point x is explicit : $\mathcal{O}(x) = \{n^k x + w, k \in \mathbb{Z}, w \in \mathbb{Z}[\frac{1}{n}]\}$.

All orbits are dense except the orbit of the global fixed point ∞.

Remark 3.1. *Applying the change of coordinate $x = \tan(\frac{u}{2})$ the standard $GA(1,\mathbb{R})$-action is given by :*

$$\Phi^{stand}_{(a,b)} : \begin{cases} [-\pi,\pi]/(-\pi \sim \pi) & \to [-\pi,\pi]/(-\pi \sim \pi) \\ u & \mapsto 2\arctan(a\tan(\frac{u}{2}) + b) \end{cases}$$

Non faithful actions.

A family of non faithful action is given by :

$$\Phi^{deg} : \begin{cases} GA(1,\mathbb{R}) & \to Diff^{\omega}_{+}(S^1) \\ (a,b) & \mapsto \varphi_{\ln a} \end{cases},$$

where (φ_t) is any flow on the circle.

The induced $BS(1,n)$-actions are the actions generated by $f(\theta) = \theta$ and $h(\theta) = \varphi_{\ln n}(\theta)$.

Remark 3.2. *There exist actions that do not come from actions of the affine group of the real line: There exist (even orientation preserving) circle homeomorphisms which do not embed in a continuous flow (see* [Zdu85]*). However, the family $< f(\theta) = \theta, h(\theta) = \varphi_{\ln n}(\theta) >$, where φ is a flow, extends to actions $< f(\theta) = \theta, h(\theta) = k(\theta) >$, where k is any circle homeomorphism.*

It is easy to see that $h \circ f \circ h^{-1} = f^n$ and that these actions are not faithful and have the dynamics of k : $\mathcal{O}(x) = \{k^n(x), n \in \mathbb{Z}\}$.

3.3.2. **Actions of $GA(1,\mathbb{R})$ and induced $BS(1,n)$ on the 2-torus.**

Taking the product of the standard action with a non faithful action of $GA(1,\mathbb{R})$ on the circle, we get a family of faithful $GA(1,\mathbb{R})$-actions on the 2-torus :

$$\Phi^{\varphi} : \begin{cases} GA(1,\mathbb{R}) & \to Diff^{\omega}_{+}(\mathbb{T}^2) \\ (a,b) & \mapsto \Phi^{\varphi}_{(a,b)} \end{cases},$$

where

$$\Phi^{\varphi}_{(a,b)} : \begin{cases} (\mathbb{R}\cup\{\infty\}) \times S^1 & \to (\mathbb{R}\cup\{\infty\}) \times S^1 \\ (x,\theta) & \mapsto (ax+b, \varphi_{\ln a}(\theta)) \end{cases}$$

and (φ_t) is any flow on the circle.

The "extended" induced $BS(1,n)$-actions are the actions generated by $f_0(x,\theta) = (x+1,\theta)$ and $h_k(x,\theta) = (nx, k(\theta))$, where k is any circle orientation preserving homeomorphism. They are faithful since they are products of two actions, and one of them is faithful. Their dynamics depend on k.

Definition 3.3.

The **standard action of** $GA(1,\mathbb{R})$ **on the 2-torus** *is the action* Φ^{φ}*, where* $\varphi_t(\theta) = \theta + t$ *is the flow of the circle rotations, that is given by*

$$\Phi^{stand}_{(a,b)} : \begin{cases} (\mathbb{R} \cup \{\infty\}) \times S^1 & \to (\mathbb{R} \cup \{\infty\}) \times S^1 \\ (x,\theta) & \mapsto (ax+b, \theta + \ln a) \end{cases}$$

The **standard action of** $BS(1,n)$ **on the 2-torus** *is the induced action, that is the action generated by* $f_0(x,\theta) = (x+1,\theta)$ *and* $h_0(x,\theta) = (nx, \theta + \ln n)$*.*

This $GA(1,\mathbb{R})$-action has no global fixed point and it has an 1-dimension circular orbit $\{\infty\} \times S^1$.

This $BS(1,n)$-action has no finite orbit, the restriction of h_0 to $\infty \times S^1$ is the irrational rotation by $\ln n$. The unique minimal set is $\infty \times S^1$.

3.4. Proof of Theorem 1 and Corollary 1.

Proof of Theorem 1.

We considered actions on the 2-torus generated by $f_0(x,\theta) = (x+1,\theta)$ and $h_k(x,\theta) = (nx, k(\theta))$.

In the previous section, we have seen that these actions are faithful $BS(1,n)$-actions, since they are products of two $BS(1,n)$-actions on S^1, where one of them is faithful.

Note that the set of f_0-fixed points, $fix(f_0)$ is the circle $\mathcal{C}_1 := \infty \times S^1$ and any horizontal circle $(\mathbb{R} \cup \{\infty\}) \times \theta_0$ is f_0-invariant. The circles $\mathcal{C}_1 = \infty \times S^1$ and $\mathcal{C}_2 := 0 \times S^1$ are h_k-invariant. The restriction of h_k to these circles is the homeomorphism k.

- If the rotation number of k is rational, then there exists a point in $\infty \times S^1$ with a h_k-finite orbit. As it is f-fixed, its BS-orbit is finite.
- If the rotation number of k is irrational (for example for the standard action), there are neither fixed points nor periodic points of h_k. Therefore there is no global fixed point for this action. Moreover, there is no finite orbit. The circle $\mathcal{C}_1$ contains the ω_{h_k}-limit set of any point in $\mathbb{T}^2 \setminus \mathcal{C}_2$ and $\mathcal{C}_2$ contains the α_{h_k}-limit set of any point in $\mathbb{T}^2 \setminus \mathcal{C}_1$. Hence, the unique minimal set M for this action is contained $\mathcal{C}_1$.

  If k is minimal, M coincides with $\mathcal{C}_1$, the set of f_0-fixed points.

  If k is a Denjoy homeomorphism, M is strictly contained in $\mathcal{C}_1$, the set of f_0-fixed points.

Proof of Corollary 1.

Consider BS-actions generated by f_0 and h_ϵ given by $h_\epsilon(x,\theta) = (nx, \theta + \ln(n) + \epsilon)$. If $\ln(n) + \epsilon$ is rational, then the restriction of h_ϵ to $\infty \times S^1$ is of finite order and every point in $\infty \times S^1$ has a finite BS-orbit, this action is clearly not topologically conjugate to the standard one. But, this can occur with ϵ arbitrary small, so for a BS-action arbitrary C^∞-close to the standard action.

3.5. Other examples of actions of $BS(1,n)$.

In this part, we construct diffeomorphisms f, h [resp. F and H] generating a faithful $BS(1,n)$ action on the circle [resp. the torus] where f [resp. F] has not fixed points but it has periodic points.

3.5.1. *On the circle.* Let us denote $< \bar{f}_i, \bar{h}_i >$ the renormalization to $[\frac{i}{n}, \frac{i+1}{n}]$ of the standard $BS(1,n)$-action on $\mathbb{R} \cup \{\infty\}$, where $i \in \{0, ..., n-1\}$.

We define $\hat{f} : [0,1]/_{(0\sim 1)} \to [0,1]/_{(0\sim 1)}$ by $\hat{f}(x) = \bar{f}_i(x)$, if $x \in [\frac{i}{n}, \frac{i+1}{n}]$ and analogously $h : [0,1]/_{(0\sim 1)} \to [0,1]/_{(0\sim 1)}$ by $h(x) = \bar{h}_i(x)$, if $x \in [\frac{i}{n}, \frac{i+1}{n}]$. It is easy to see that the group generated by $\hat{f}$ and h is isomorphic to $BS(1,n)$.

Let $f = R_{\frac{1}{n-1}} \circ \hat{f}$, where $R_{\frac{1}{n-1}}(x) = x + \frac{1}{n-1}(mod 1)$.

We claim that the group generated by f and h is isomorphic to $BS(1,n)$.

More precisely, $h \circ f \circ h^{-1} = h \circ R_{\frac{1}{n-1}} \circ \hat{f} \circ h^{-1} = R_{\frac{1}{n-1}} \circ h \circ \hat{f} \circ h^{-1}$ since by construction $R_{\frac{1}{n-1}}$ commutes with h (and also with $\hat{f}$).

Then $h \circ f \circ h^{-1} = R_{\frac{1}{n-1}} \circ \hat{f}^n = (R_{\frac{1}{n-1}} \circ \hat{f})^n = f^n$ since $R_{\frac{1}{n-1}}$ commutes with $\hat{f}$ and has order $n-1$. Hence, f and h generate an action of $BS(1,n)$.

This action is faithful, since it is a well known fact that for a non faithful action of $BS(1,n)$, f has finite order. By construction f admits exactly $n-1$ periodic points of period $n-1$, so f is not of finite order.

This construction provides an example of two circle diffeomorphisms f and h generating a faithful action of $BS(1,n)$, where f has no fixed points but periodic ones.

3.5.2. *On the Torus.* Let f and h be the circle diffeomorphisms as below. We define two torus diffeomorphisms :

$$F : \begin{cases} (\mathbb{R} \cup \{\infty\}) \times [0,1]/_{(0\sim 1)} & \to (\mathbb{R} \cup \{\infty\}) \times [0,1]/_{(0\sim 1)} \\ (x,y) & \mapsto (x+1, f(y)) \end{cases}$$

and

$$H : \begin{cases} (\mathbb{R} \cup \{\infty\}) \times [0,1]/_{(0\sim 1)} & \to (\mathbb{R} \cup \{\infty\}) \times [0,1]/_{(0\sim 1)} \\ (x,y) & \mapsto (nx, h(y)) \end{cases}.$$

The diffeomorphisms F and H generate a faithful action of $BS(1,n)$ on the torus, F admits periodic points but not fixed points.

4. Isotopy class of f and rotation set.

The aim of this section is proving Theorem 3.

Proposition 4.1. *Let $< f, h >$ be a faithful representation of $BS(1,n)$ on $\mathbb{T}^2$. There exists a positive integer N ($N \in \{1,2,3,4,6\}$) such that f^N is isotopic to identity.*

Proof. For proving the proposition, it is enough to prove that there exists $N \in \mathbb{N}$ such that $A_f^N = Id$.

As $A_f \in GL(2,\mathbb{Z})$ and f is conjugated to f^n we have :

- the linear maps A_f and $A_{f^n} = A_f^n$ are conjugated by $A_h \in GL(2,\mathbb{Z})$,
- the modulus of the eigenvalues of A_f are 1.
- the product of the eigenvalues is $+1$ or -1.
- the trace of A_f is an integer.

Case 1: A_f admits a real eigenvalue.

In this case, the possible eigenvalues are $+1$ or -1 and A_f is conjugated to one of the following applications:

$$A_1 = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \qquad \text{or} \qquad A_2 = \begin{pmatrix} \varepsilon_1 & 1 \\ 0 & \varepsilon_1 \end{pmatrix}$$

where $\varepsilon_i \in \{-1, 1\}$, $i = 1, 2$.

It is clear that $A_1^2 = Id$. We are going to prove that A_2 cannot occur.

If $\varepsilon_1 = 1$ then $A_2^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. One can see that A_2^n can not be conjugated to A_2 in $GL(2, \mathbb{Z})$. More precisely, one can compute the conjugating matrix in $GL(2, \mathbb{R})$, it is of the form : $A_h^{-1} = \begin{pmatrix} \frac{1}{\sqrt{n}} & b \\ 0 & \sqrt{n} \end{pmatrix}$ where $b \in \mathbb{Z}$. This matrix does not belong to $GL(2, \mathbb{Z})$. This case is not possible.

If $\varepsilon_1 = -1$ then $A_2^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$, if n is even or $A_2^n = \begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$, if n is odd.

For n even, one can easily see that A_2^n can not be conjugated to A_2, since $Tr(A_2^n) = 2 \neq Tr(A_2)$.

For n odd, one can see that A_2^n can not be conjugated to A_2 in $GL(2, \mathbb{Z})$: one compute the conjugating matrix in $GL(2, \mathbb{R})$, it is of the form: $A_h = \begin{pmatrix} \sqrt{n} & b \\ 0 & \frac{1}{\sqrt{n}} \end{pmatrix}$ where $b \in \mathbb{Z}$. This matrix does not belong to $GL(2, \mathbb{Z})$. This case is not possible.

Case 2: A_f has complex eigenvalues.

Necessary A_f has two eigenvalues λ, $\bar{\lambda}$. Moreover $|\lambda| = 1$. So $det A_f = \lambda\bar{\lambda} = |\lambda|^2 = 1$.

Hence, A_f is conjugated to a rotation of angle θ. The trace of A_f is $2\cos\theta$ and it is an integer. Then the possible values for $\cos\theta$ are : $0, 1, -1, \frac{1}{2}, -\frac{1}{2}$.

If $\cos\theta \in \{1, -1\}$ then $A_f = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$, where $\varepsilon \in \{-1, 1\}$.

If $\cos\theta = 0$ then A_f is conjugated to $\begin{pmatrix} 0 & \varepsilon_1 \\ -\varepsilon_1 & 0 \end{pmatrix}$ (where $\varepsilon_1 \in \{-1, 1\}$) which is of order 4, so $A_f^4 = Id$.

If $\cos\theta \in \{\frac{1}{2}, -\frac{1}{2}\}$ then A_f is conjugated to the rotation of angle $l\frac{\pi}{3}$, where $l \in \{1, 2, 4, 5\}$. Therefore, $A_f^6 = Id$.

□

According to the previous proposition, given an action of $BS(1, n) =< f, h >$ on $\mathbb{T}^2$ there exists an integer N such that f^N is isotopic to identity. From now on, we assume that f is isotopic to identity : this is not a restrictive hypothesis since the action of $< f^N, h >$ on $\mathbb{T}^2$ is an action of the Baumslag-Solitar group $BS(1, nN)$.

For proving Theorem 3, we begin by proving the following:

Proposition 4.2. *If f is isotopic to identity and $\tilde{f}$ is a lift of f, then $\rho(\tilde{f})$ is a rational point.*

For proving this proposition we need the following lemmas:

Lemma 4.1. *Let* $H \in Homeo_{\mathbb{Z}^2}(\mathbb{R}^2)$ *and* $F \in Homeo^0_{\mathbb{Z}^2}(\mathbb{R}^2)$ *then* $\rho(H \circ F \circ H^{-1}) = A_H(\rho(F))$.

Proof

Case 1: $A_H = Id$**.** We prove that $\rho(H \circ F \circ H^{-1}) = \rho(F)$.

Let (a, b) be a vector in the rotation set of $H \circ F \circ H^{-1}$. By definition,

$$(a, b) = \lim_{i\to\infty} \frac{(H \circ F \circ H^{-1})^{n_i}(\widetilde{x}_i) - \widetilde{x}_i}{n_i}.$$

Then $(a, b) = \lim_{i\to\infty} \dfrac{H \circ F^{n_i} \circ H^{-1}(\widetilde{x}_i) - \widetilde{x}_i}{n_i} =$

$$= \lim_{i\to\infty} \frac{H \circ F^{n_i} \circ H^{-1}(\widetilde{x}_i) - F^{n_i} \circ H^{-1}(\widetilde{x}_i)}{n_i} + \frac{F^{n_i} \circ H^{-1}(\widetilde{x}_i) - \widetilde{x}_i}{n_i} =$$

$$= \lim_{i\to\infty} \frac{(H - Id)(F^{n_i} \circ H^{-1}(\widetilde{x}_i))}{n_i} + \frac{F^{n_i} \circ H^{-1}(\widetilde{x}_i) - \widetilde{x}_i}{n_i}.$$

As $A_H = Id$, the map $(H - Id)$ is bounded (periodic) so the limit:

$$\lim_{i\to\infty} \frac{(H - Id)(F^{n_i} \circ H^{-1}(\widetilde{x}_i))}{n_i} = (0, 0).$$

Moreover,

$$\lim_{i\to\infty} \frac{F^{n_i} \circ H^{-1}(\widetilde{x}_i) - \widetilde{x}_i}{n_i} = \lim_{i\to\infty} \frac{F^{n_i} \circ H^{-1}(\widetilde{x}_i) - H^{-1}(\widetilde{x}_i)}{n_i} + \frac{H^{-1}(\widetilde{x}_i) - \widetilde{x}_i}{n_i}.$$

By definition of the rotation set, the limit :

$$\lim_{i\to\infty} \frac{F^{n_i} \circ H^{-1}(\widetilde{x}_i) - H^{-1}(\widetilde{x}_i)}{n_i}$$

belongs to $\rho(F)$.

As $A_{H^{-1}} = Id$, the map $(H^{-1} - Id)$ is bounded so the limit :

$$\lim_{i\to\infty} \frac{(H^{-1} - Id)(\widetilde{x}_i)}{n_i} = (0, 0).$$

Finally, $(a, b) \in \rho(F)$. This proves the inclusion $\rho(H \circ F \circ H^{-1}) \subset \rho(F)$. By writing this inclusion with H^{-1} instead of H and $H \circ F \circ H^{-1}$ instead of F we obtain : $\rho(H^{-1} \circ (H \circ F \circ H^{-1}) \circ H) \subset \rho(H \circ F \circ H^{-1})$ that is $\rho(F) \subset \rho(H \circ F \circ H^{-1})$.

Case 2: H is a linear map that is $H = A_H$. We prove that $\rho(H \circ F \circ H^{-1}) = A_H(\rho(F))$.

Let (a, b) be a vector in the rotation set of $H \circ F \circ H^{-1}$. By definition,

$$(a, b) = \lim_{i\to\infty} \frac{H \circ F^{n_i} \circ H^{-1}(\widetilde{x}_i) - \widetilde{x}_i}{n_i} = \lim_{i\to\infty} \frac{A_H \circ F^{n_i} \circ A_H^{-1}(\widetilde{x}_i) - \widetilde{x}_i}{n_i} =$$

$$= \lim_{i\to\infty} \frac{A_H \left(F^{n_i} \circ A_H^{-1}(\widetilde{x}_i) - A_H^{-1}(\widetilde{x}_i)\right)}{n_i} = A_H \left(\lim_{i\to\infty} \frac{F^{n_i} \circ A_H^{-1}(\widetilde{x}_i) - A_H^{-1}(\widetilde{x}_i)}{n_i} \right) \in A_H(\rho(F)).$$

This proves the inclusion $\rho(H \circ F \circ H^{-1}) \subset A_H(\rho(F))$. We obtain the other inclusion with analogous arguments as in case 1.

Case 3: General case.
We claim that the map $A_H^{-1} \circ H \in Homeo^0_{\mathbb{Z}^2}(\mathbb{R}^2)$:
Let P an integer vector in $\mathbb{R}^2$ and x be a point of $\mathbb{R}^2$.
$A_H^{-1} \circ H(x+P) = A_H^{-1}(H(x) + A_H(P)) = A_H^{-1} \circ H(x) + A_H^{-1} \circ A_H(P) = A_H^{-1} \circ H(x) + P$.
By case 1, we have $\rho(A_H^{-1} \circ H \circ F \circ H^{-1} \circ A_H) = \rho(F)$.
By case 2, we have $\rho(A_H^{-1} \circ H \circ F \circ H^{-1} \circ A_H) = A_H^{-1}(\rho(H \circ F \circ H^{-1}))$.
Then $\rho(F) = A_H^{-1}(\rho(H \circ F \circ H^{-1}))$, it follows that $\rho(H \circ F \circ H^{-1}) = A_H(\rho(F))$. □

Lemma 4.2. *Let $< f, h >$ be a faithful representation of $BS(1,n)$ on $\mathbb{T}^2$ such that f is isotopic to identity. Then $\rho(\tilde{f}) = \frac{1}{n}(\tau_Q \circ A_{\tilde{h}})(\rho(\tilde{f}))$, where Q is an integer vector in $\mathbb{R}^2$ and τ_Q denotes the translation of vector Q.*

Proof. As two lifts of a torus map differ by an integer vector, we have that
$\widetilde{h \circ f \circ h^{-1}} = \widetilde{h} \circ \widetilde{f} \circ \widetilde{h}^{-1} + P$, for some integer vector P. By iterating this formula we have:
$\widetilde{h \circ f^k \circ h^{-1}} = \widetilde{h} \circ \widetilde{f}^k \circ \widetilde{h}^{-1} + kP$. Then

$$\frac{\widetilde{h \circ f^k \circ h^{-1}}}{k} = \frac{\widetilde{h} \circ \widetilde{f}^k \circ \widetilde{h}^{-1}}{k} + P$$

Hence, by properties of the rotation set we have $\rho(\widetilde{h \circ f \circ h^{-1}}) = \rho(\widetilde{h} \circ \widetilde{f} \circ \widetilde{h}^{-1}) + P = A_{\tilde{h}}(\rho(\tilde{f})) + P$, because of the previous lemma.
Since $f^n = h \circ f \circ h^{-1}$, we have $\rho(\widetilde{f^n}) = A_{\tilde{h}}(\rho(\tilde{f})) + P$.
Since $(\widetilde{f^n})$ and $(\tilde{f})^n$ are two lifts of f^n, then $n\rho(\widetilde{f}) + P' = A_{\tilde{h}}(\rho(\tilde{f})) + P$, for some integer vector P'.
Finally, $\rho(\widetilde{f}) = \frac{1}{n}(\tau_Q \circ A_{\tilde{h}})(\rho(\tilde{f}))$, for some integer vector Q. □

Proof of the Proposition 4.2.
Let B be the affine map of $\mathbb{R}^2$ given by $B = \frac{1}{n}(\tau_Q \circ A_{\tilde{h}})$. Note that the linear part $\bar{B}$ of B satisfies $det\bar{B} = \frac{1}{n^2} det A_{\tilde{h}} = \pm\frac{1}{n^2}$. The formula given by Lemma 4.2 can be written as $\rho(\widetilde{f}) = B(\rho(\widetilde{f}))$. By taking the volumes, we get: $vol(\rho(\widetilde{f})) = |det\bar{B}| vol(\rho(\widetilde{f})) = \frac{1}{n^2} vol(\rho(\widetilde{f}))$. Then $vol(\rho(\widetilde{f})) = 0$ since $\rho(\widetilde{f})$ is a compact set.
This implies that $\rho(\widetilde{f})$ has empty interior, so since it is a convex set, it is either a segment or a point.
In the case where $\rho(\widetilde{f})$ is a segment $[a, c]$ with $a \neq c$, since $B([a, c]) = [a, c]$ we either have $B(a) = a$ and $B(c) = c$ or $B(a) = c$ and $B(c) = a$ in both cases $B^2(a) = a$ and $B^2(c) = c$.
As B^2 is an affine map, its linear part has 1 as eigenvalue, its trace is $\frac{1}{n^2} Tr(A^2_{\tilde{h}})$ so it has the form $\frac{p}{n^2}$ with $p \in \mathbb{Z}$.
Its determinant is $\frac{1}{n^4}$ so its other eigenvalue is $\frac{1}{n^4}$.
Therefore its trace is $1 + \frac{1}{n^4}$ so has not the form $\frac{p}{n^2}$ with $p \in \mathbb{Z}$, this is a contradiction.

Consequently, the rotation set $\rho(\widetilde{f})$ is a single point which is the unique fixed point of the affine map B. Since B has rational coefficients, then $\rho(\widetilde{f})$ has rational coordinates.

□

Proof of the Theorem 3.

According to Proposition 4.1, there is an integer N such that f^N is isotopic to identity. By Proposition 4.2, the rotation number of any lift $\widetilde{f}^N$ is a rational vector.

Let us write $\rho(\widetilde{f}^N) = (\frac{p_1}{q}, \frac{p_2}{q})$, where p_1, p_2, q are integers. Hence, $\rho(\widetilde{f}^{Nq}) = (p_1, p_2) \in \mathbb{Z}^2$, then there is a lift of f^{Nq} which has rotation set equal to $\{(0,0)\}$.

According to Corollary 3.5 of [MZ89], $\{(0,0)\} = \rho(\widetilde{f^{Nq}}) = Conv(\rho_{erg}(\widetilde{f^{Nq}}))$, where $\rho_{erg}(\widetilde{f}) := \{\int(\widetilde{f} - id)d\mu,$ where μ is an ergodic f-invariant measure$\}$.

Hence $\{(0,0)\} = \rho_{erg}(\widetilde{f^{Nq}})$. Then, using Theorem 3.5 of [Fra89], f^{Nq} has a fixed point and therefore $fix(f^{Nq})$ is non empty.

□

5. Existence of a BS-minimal set in $per(f)$.

The aim of this section is to show the existence of a minimal set for the action included in the set of f-periodic points. In the case that $< f, h >$ is a representation of $BS(1,n)$ in $Homeo(X)$, where X is a compact metric space (Theorem 4), we ask for the existence of fixed or periodic points of f and in Theorem 5 we assume that any f-invariant probability measure has support included in the set of f-fixed points. In this case, we also study f-minimal sets and the topological entropy of f. In this section we also prove Corollaries 2 and 3.

We are going to prove Theorem 4.

Proof of (a)(1).

Let $x_0 \in fix(f)$. Since $h^j \circ f \circ h^{-j}(x_0) = f^{n^j}(x_0) = x_0$ for any $j \in \mathbb{N}$, then $f \circ h^{-j}(x_0) = h^{-j}(x_0)$ so $h^{-j}(x_0) \in fix(f)$ and the α-limit set of x_0 for h is included in $fix(f)$.

Proof of (a)(2).

Let M be an f-invariant set, then $f(h^{-1}(M)) = h^{-1}(f^n(M)) \subset h^{-1}(M)$. Since $h^j \circ f \circ h^{-j} = f^{n^j}$ for any $j \in \mathbb{N}$, then $h^{-j}(M)$ is f-invariant.

Let $P = fix(f)$. It holds that $h^{-1}(P) \subset P$ so $h^{-j}(P)$ is a closed f-invariant set for any $j \in \mathbb{N}$.

Let $K_n = \bigcap_{-n}^{n} h^{-l}(P)$, since $fix(f) \neq \emptyset$, then $\{K_n\}$ is a family of decreasing f-invariant non-empty closed sets, therefore $K = \bigcap_{-\infty}^{\infty} h^{-l}(P)$ is a closed non-empty set invariant by f and h. As a consequence there exists a BS-minimal set included in K.

Let us call $M_{BS} \subset fix(f)$ a minimal set for the group. Since M_{BS} is h-invariant there exists an h minimal set $M_h \subset M_{BS}$. The set M_h is also f-invariant (it is contained in $fix(f)$), so it follows that $M_h = M_{BS}$.

Proof of (a)(3).

Since $fix(f) \subset fix(f^n)$ and $h(fix(f)) = fix(f^n)$ then $\sharp\{fix(f)\} = \sharp\{fix(f^n)\}$. Therefore $fix(f) = h(fix(f))$ and h has a periodic point in $fix(f)$.

Proof of (a)(4).
Let $\mathcal{M}$ be a BS-minimal set verifying $\mathcal{M} \cap fix(f) \neq \emptyset$. Let $x \in \mathcal{M} \cap fix(f)$ then $\alpha_h(x)$, the α-limit set of x for h, is a h-invariant closed set verifying $\alpha_h(x) \subset fix(f)$. Let $M_x \subset \alpha_h(x)$ be a minimal set for the group. Since $x \in \mathcal{M}$ then $\alpha_h(x) \subset \mathcal{M}$, therefore $M_x = \mathcal{M}$ and the claim follows.

Proof of (b).
Suppose that $per(f) \neq \emptyset$, then there exists an positive integer N such that $fixf^N \neq \emptyset$.
According to item (a)(2), there is a minimal set M_N of $< f^N, h >$ such that $M_N \subset fixf^N$.
Let $\mathcal{M} = \cup_{k=0}^{N-1} f^k(M_N)$.

- $\mathcal{M}$ is f-invariant : $f(\mathcal{M}) = \cup_{k=0}^{N-1} f^{k+1}(M_N) = \mathcal{M}$, since $f^N(M_N) = M_N$. In fact, $\mathcal{M} = \cup_{k\in\mathbb{Z}} f^k(M_N)$.
- $\mathcal{M}$ is h-invariant : $h(\mathcal{M}) = \cup_{k=0}^{N-1} h \circ f^k(M_N) = \cup_{k=0}^{N-1} f^{nk} \circ h(M_N) = \cup_{k=0}^{N-1} f^{nk}(M_N) \subset \mathcal{M}$, since M_N is h-invariant.

Since $\mathcal{M}$ is closed, non empty, f and h invariant, it contains a BS minimal set M.

- $\mathcal{M} \subset fix(f^N)$: let $x \in \mathcal{M}$, there is a $k = 0, ..., N-1$ and $x' \in M_N$ such that $x = f^k(x')$. Hence $f^N(x) = f^{N+k}(x') = f^k(f^N(x')) = f^k(x') = x$.

Finally, $M \subset \mathcal{M} \subset fix(f^N)$.

□

Proof of Corollary 2.

If the action were minimal, its unique minimal set would be X and would be contained in $fixf^N$, according to item (b) of Theorem 4. This implies that $X = fixf^N$ and so $f^N = Id$, the action would not be faithful. This is a contradiction. □

The following is the proof of Theorem 5.

Proof of (1). Let M_f be an f-minimal set and $x_0 \in M_f$. Let $\mu_k = \dfrac{1}{k}\sum_{i=0}^{k-1} \delta_{f^i(x_0)}$ and μ a weak limit of μ_k. It is known that μ is an f-invariant probability measure and its support is included in $M_f = \overline{\mathcal{O}_f(x_0)}$, the closure of the f-orbit of x_0. In addition, by hypotheses $supp(\mu) \subseteq fix(f)$ then $M_f \cap fix(f) \neq \emptyset$. It follows that M_f is reduced to a fixed point. Since a periodic orbit is a minimal set we have that $per(f)$ coincides with the set $fix(f)$.

Proof of (2). Let M_{BS} be an BS-minimal set. Since M_{BS} is f-invariant, there exists an f minimal set $M_f \subset M_{BS}$. Since M_f is an f-fixed point, it follows that $M_{BS} \cap fix(f) \neq \emptyset$, so according to items (a)(2) and (a)(4) of Theorem 4, $M_{BS} \subseteq fix(f)$ and it coincides with a h-minimal set in $fix(f)$.

Proof of (3). Recall that $ent_{top}(f) = sup\{ent_\nu(f)\}$ where ν is an f-invariant probability measure, the supremum of all metric entropies. Since $ent_\nu(f) = ent_\nu(f|_{supp(\nu)})$ and $f|_{supp(\nu)} = Id$, we have that $ent_{top}(f) = 0$.

□

We finish this section by proving Corollary 3.

If $S = \mathbb{T}^2$, let N be the positive integer given by Theorem 1, then the set $fix(f^N) \neq \emptyset$. If $S = S^2$, let N be the smallest positive integer such that f^N has at least three fixed points and it is orientation preserving. Otherwise, $N = 1$.

In addition, f is distortion element of $BS(1,n)$, so according to Theorem 1.3 of [FH06] for any f-invariant probability measure, μ, it holds that $supp(\mu) \subseteq fix(f^N)$ so Theorem 5 implies the claim of this corollary except the f-ellipticity of the points in minimal sets.

For simplicity, we will prove the ellipticity in the case where f has fixed points. The general case is analogous.

Let x_0 be a point in a BS-minimal set M_{BS}. Since M_{BS} is also an h-minimal set, the h-orbit of x_0 is recurrent. Then there exists a subsequence (n_k) $(n_k \to \infty)$ such that $h^{-n_k}(x_0) \to x_0$.

From $h^{n_k} \circ f \circ h^{-n_k} = f^{n^{n_k}}$, we deduce that :

$$Dh^{n_k}(f(h^{-n_k}(x_0))) \circ Df(h^{-n_k}(x_0)) \circ Dh^{-n_k}(x_0) = Df^{n^{n_k}}(x_0).$$

As the points x_0 and $h^{-n_k}(x_0)$ are fixed by f and $(Dh^{-n_k}(x_0))^{-1} = Dh^{n_k}(h^{-n_k}(x_0))$, then:

$$(Dh^{-n_k}(x_0))^{-1} \circ Df(h^{-n_k}(x_0)) \circ Dh^{-n_k}(x_0) = (Df(x_0))^{n^{n_k}}.$$

So $Df(h^{-n_k}(x_0))$ and $(Df(x_0))^{n^{n_k}}$ have the same eigenvalues. As f is C^1 and $h^{-n_k}(x_0) \to x_0$, then $Df(h^{-n_k}(x_0)) \to Df(x_0)$.

We conclude that $Df(x_0)$ and $(Df(x_0))^{n^{n_k}}$ have the same eigenvalues, finally the eigenvalues of $Df(x_0)$ have module 1.

□

6. Perturbations of the standard $BS(1,n)-$action on $\mathbb{T}^2$.

Let us recall that :

- the standard BS-action on $\mathbb{T}^2$ is the one generated by the two diffeomorphisms of $\mathbb{R} \cup \{\infty\} \times S^1 : f_0(x,\theta) = (x+1,\theta)$ and $h_0(x,\theta) = (nx, \theta + \ln(n))$,
- $\mathcal{C}_1 := \infty \times S^1$ and $\mathcal{C}_2 := 0 \times S^1$.

Before proving Theorem 2, we prove the following

Lemma 6.1. *Let us consider a BS-action $< f, h >$ on $\mathbb{T}^2$ generated by f and h sufficiently C^0-close to homeomorphisms $\bar{f}_0$ and $\bar{h}_0$ which generate a BS-action. If both $\bar{f}_0$ and $\bar{h}_0$ are isotopic to identity and the rotation set of a lift of $\bar{f}_0$ is $(0,0)$, then the rotation set of a lift of f is $(0,0)$.*

Proof of the lemma.

For (f,h) sufficiently close to $(\bar{f}_0, \bar{h}_0)$, f and h are isotopic to identity. By Lemma 4.2 the rotation set of any lift $\tilde{f}$ of f satisfy $n\rho(\tilde{f}) = \rho(\tilde{f}) + (p,q)$, where (p,q) is an integer vector. Then the rotation set of $\tilde{f}$ is a rational vector $(\frac{p}{n-1}, \frac{q}{n-1})$, with p, q integers.

It is proved in [MZ89] that the rotation set map $\rho : Homeo_{\mathbb{Z}^2}(\mathbb{R}^2) \to \mathcal{K}(\mathbb{R}^2)$, the set of compact subsets of $\mathbb{R}^2$ is upper semi-continuous with respect to the compact-open topology on $Homeo_{\mathbb{Z}^2}(\mathbb{R}^2)$ and the Haussdorff topology on $\mathcal{K}(\mathbb{R}^2)$. In other words, if G is

an element of $Homeo_{\mathbb{Z}^2}(\mathbb{R}^2)$ and U is a neighborhood of $\rho(G)$ in $\mathbb{R}^2$, then for F sufficiently close to G, we have $\rho(F) \subset U$.

The rotation set of a lift of $\bar{f}_0$ is $(0,0)$, consider a neighborhood U of $(0,0)$ in $\mathbb{R}^2$ that contains no points of the form $(\frac{p}{n-1}, \frac{q}{n-1})$, with p, q integers and $(p,q) \neq (0,0)$.

According to the previous result of [MZ89], for f sufficiently close to f_0, the rotation set of a lift of f is included in U and since it has form $(\frac{p}{n-1}, \frac{q}{n-1})$, it must be $(0,0)$. $\square$

Proof of Theorem 2.

(1) The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are h_0-normally hyperbolic in the sense of [HPS77]. Consider a neighborhood U_1 of $\mathcal{C}_1$ where $\mathcal{C}_1$ is h_0-attractive and a neighborhood U_2 of $\mathcal{C}_2$ where $\mathcal{C}_2$ is h_0-repulsive. Obviously, there exists some integer k_0 such that $h_0^k(\mathbb{T}^2 \setminus U_2) \subset U_1$, $h_0^{-k}(\mathbb{T}^2 \setminus U_1) \subset U_2$, for all $k \geq k_0$.

According to Theorem 4.1 of [HPS77], there exists a C^1-neighborhood $\mathcal{V}$ of h_0 in $Diff^1(\mathbb{T}^2)$ such that for all $h \in \mathcal{V}$ there exist two circles $\mathcal{C}_1'$ and $\mathcal{C}_2'$ which are C^1-closed to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively and they are h-invariant.

Moreover $\mathcal{C}_1'$ is h-attractive in U_1 and $\mathcal{C}_2'$ is h-repulsive in U_2 and $h^k(\mathbb{T}^2 \setminus U_2) \subset U_1$, $h^{-k}(\mathbb{T}^2 \setminus U_1) \subset U_2$, for all $k \geq k_0$. Therefore, item (1) is proved.

(2) Obviously, the rotation set of a lift of f_0 is $(0,0)$. According to previous lemma, the rotation set of a lift of f must be $(0,0)$. Then $fix(f)$ is not empty.

Let $x_0 \in fix(f)$, if $x_0 \notin \mathcal{C}_1'$ the its α-limit set for h is included in $\mathcal{C}_2'$ and consists of f-fixed points, according to Theorem 4. In other words, $\mathcal{C}_2'$ intersects $fix(f)$. But for f sufficiently close to f_0 we have that $f^j(\mathcal{C}_2') \cap \mathcal{C}_2' = \emptyset$, for any $j \neq 0$. Hence $x_0 \in \mathcal{C}_1'$.

(3) We first prove that any minimal set of BS intersects $\mathcal{C}_1'$.

Let us consider M a BS-minimal set :

Suppose that $M \subset \mathcal{C}_2'$ then $f(M) = M \subset f(\mathcal{C}_2')$ then $f(\mathcal{C}_2') \cap \mathcal{C}_2' \neq \emptyset$ which is contradiction for f close to f_0.

Since $M \not\subset \mathcal{C}_2'$, there is $x_0 \in M \setminus \mathcal{C}_2'$. Then $\omega_h(x_0) \subset \mathcal{C}_1' \cap M$, so we are done.

The circle $\mathcal{C}_1'$ is h-invariant, we can consider the rotation number ρ of the restriction of h to $\mathcal{C}_1'$:

Case 1: $\rho \in \mathbb{Q}$**.**

There is a BS-minimal set $\mathcal{M}$ included in $fix(f)$ so in $\mathcal{C}_1'$. This set $\mathcal{M}$ contains an h-minimal set in $\mathcal{C}_1'$. Moreover h has periodic orbit and any minimal set of $h_{|\mathcal{C}_1'}$ is an h-periodic orbit. Then there is an h-periodic orbit contained in $\mathcal{M} \subset fix(f)$. So this h-periodic orbit is a finite BS-orbit.

Case 2: $\rho \notin \mathbb{Q}$**.**

Case 2a: $h_{|\mathcal{C}_1'}$ **is conjugated to an irrational rotation.**

We claim that $\mathcal{C}_1' = fix(f)$. Let $x_0 \in fix(f)$, then $\alpha_h(x_0) = \mathcal{C}_1'$ and it is contained in $fix(f)$. Hence $fix(f) = \mathcal{C}_1'$.

Now, we prove that $\mathcal{C}_1'$ is a minimal set for the BS-action:

Let x be in $\mathcal{C}_1' = fixf$. The closure of h-orbit of x is BS-invariant and coincide with $\mathcal{C}_1'$. Consequently, the circle $\mathcal{C}_1'$ is a minimal set for the BS-action.

Case 2b: $h_{|\mathcal{C}_1'}$ is semi-conjugated (not conjugated) to an irrational rotation.

Then $h_{|\mathcal{C}_1'}$ admits a unique minimal set K that is homeomorphic to a Cantor set.

Let x_0 be a fixed point of f, then $K = \alpha_h(x_0) \subset fix(f)$. So K is BS-invariant, so it contains a BS-minimal set.

Since any BS-minimal set intersects $\mathcal{C}_1'$ and $\alpha_h(x) = K$ for all $x \in \mathcal{C}_1'$, any BS-minimal set contains K.

Finally, K is the unique BS-minimal set and it is a Cantor set.

□

In the case that the action is C^2 we have the following

Corollary 4. *If the action is C^2 and sufficiently C^1-close to $< f_0, h_0 >$ then either :*

(1) *$\mathcal{C}_1' = fix(f)$ is the unique minimal set for the action and the minimal sets of f are its fixed points or*
(2) *there exists a finite BS-orbit contained in $\mathcal{C}_1'$.*

Proof.

According to theorem 2 (3), either there exists a finite BS-orbit in $\mathcal{C}_1'$ or the action has an unique minimal set M which is the unique $h_{|\mathcal{C}_1'}$-minimal set.

In the second case, since h is C^2, the circle map $h_{|\mathcal{C}_1'}$ is C^2 and according to Denjoy's theorem, M is the whole circle $\mathcal{C}_1' = fix(f)$. □

6.1. **Persistent global fixed point.**

Proposition 6.1. *Let us consider a BS-action $< f, h >$ on $\mathbb{T}^2$ generated by f and h sufficiently C^1-close to $\bar{f}_0$ and $\bar{h}_0$, where $\bar{f}_0$ and $\bar{h}_0$ are isotopic to identity. If the rotation set of a lift of $\bar{f}_0$ is $(0,0)$ and $\bar{h}_0$ is a Morse Smale diffeomorphism satisfying that any periodic point is $\bar{h}_0$-fixed, then $< f, h >$ admits fixed point.*

Proof.

Any h sufficiently C^1-close to $\bar{h}_0$ is a Morse Smale diffeomorphism where any h-periodic point is fixed. In particular, any h-minimal set is an h-fixed point.

By lemma 6.1, rotation set of a lift f is $(0,0)$, so $fix(f)$ is not empty.

As a consequence of Theorem 4, there is a BS-minimal set included in $fix(f)$, this minimal set contains an h-minimal set, that is a fixed point of h. This point is a global fixed point. □

Explicit example.

Let $\bar{f}_0(x,\theta) = (x+1, \theta+1)$ and $\bar{h}_0(x,\theta) = (nx, n\theta)$, where $x \in \mathbb{R} \cup \infty$ and $\theta \in \mathbb{R} \cup \infty$. It is easy to check that both diffeomorphisms are isotopic to identity, $\bar{h}_0$ is a Morse Smale diffeomorphism with two fixed points: $(0,0)$ and (∞,∞) and $\bar{f}_0$ has a unique fixed point: (∞,∞). These diffeomorphisms satisfy the hypothesis of the previous proposition, so any sufficiently C^1-close BS- action has fixed point.

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